Pergamon

0021-8928(95)00061-5

J. Appl. Maths Mechs, Vol. 59, No. 4, pp. 525–531, 1995
Copyright © 1996 Elsevier Science Ltd
Printed in Great Britain. All rights reserved
0021-8928/95 $24.00+0.00

ON THE THEORY OF PARTIAL STABILITY†

V. I. VOROTNIKOV

Nizhnii Tagil

(*Received* 23 *June* 1994)

The problem of stability with respect to some of the variables (partial stability) is considered. The motives and "aftereffects" of the designation of admissible boundaries within which the "uncontrollable" variables are allowed to vary in a given problem are analysed. It is shown that the imposition of additional constraints on the "uncontrollable" variables while testing for partial stability makes it possible (and desirable) to use Lyapunov functions with properties "intermediate" between the conventional ones. It is also shown how to define the notion of sign-definiteness of Lyapunov functions with respect to some of the variables in order to avoid such "intermediate" properties. The main partial stability theorem is extended in order to deal with the newly arising conditions. Certain assumptions are introduced that provide a better understanding of the performance of partially stable systems. Examples are given.

The problem of the stability of motion with respect to some of the variables [1, 2], also known as partial stability (PS), arises naturally in applications. In the most general context, the PS problem is analysed as a special kind of y-stability problem, concerning the position $\mathbf{x} = (\mathbf{y}, \mathbf{z}) = \mathbf{0}$ of a system of ordinary differential equations [2–14]

$$\mathbf{x}^* = \mathbf{X}(t, \mathbf{x}), \mathbf{X}(t, \mathbf{0}) \equiv 0 \tag{0.1}$$

In such treatments, system (0.1) is constructed (anew each time) as a system of perturbed motion in the process whose stability is under investigation. Among other problems related to the PS problem are: stability with respect to given state functions [15, 16], stability with respect to two measures [17–19], and polystability [20, 21].

The present state-of-the-art in PS theory proper and its applications can be judged from [2–14], where the reader will find information about the problems, methods and special features of the research, as well as a bibliography.

However, little attention has been paid to the motives and "aftereffects" of the designation of the admissible boundaries within which the "uncontrollable" z-variables may vary in PS problems. Yet, the question of whether the investigation of PS problems is effective depends essentially on this factor.

In addition, the special features of PS system performance deserve attention. The fact is that PS theory is concerned with rather delicate properties of the system. The necessary "persistence" of these properties depends on a larger number of factors than do the properties of "total" stability. What is required is a deeper understanding of the nature of PS problems, of the laws governing the performance of PS systems and of the mechanisms through which PS may be achieved or lost.

The purpose of this paper is to study these questions.

1. MOTIVES FOR DESIGNATING ADMISSIBLE BOUNDARIES OF VARIATION FOR "UNCONTROLLABLE" VARIABLES

When one is studying the y-stability of the position $\mathbf{x} = \mathbf{0}$ of system (0.1), the behaviour of the z-variables does not, in principle, require monitoring (provided certain general conditions are observed). In the coupled system (0.1), however, they exert an important influence on the "main" y-variables.

The factors that determine the choice of the "uncontrollable" x-variables are as follows:

1. *Allowance for the "worst case" scenario (other, general, conditions being the same) in the variation of "uncontrollable" variables.* This entails the assumption $\| \mathbf{z} \| < \infty$ and, consequently, the study of y-stability of the position $\mathbf{x} = \mathbf{0}$ of system (0.1) in the domain

$$t \geqslant 0, \| \mathbf{y} \| \leqslant H = \text{const} > 0, \| \mathbf{z} \| < \infty \tag{1.1}$$

Such considerations may prove overly cautious. Indeed, one does not utilize inequalities $| z_i | \leqslant H$ that are valid (or admissible) for certain z-components, or relations like $| f_i(t, \mathbf{x}) | \leqslant H$. Such relations may considerably facilitate testing for y-stability. In a sense, allowance for the "worst case" scenario in PS theory is comparable with the ideas of game theory [22].

†*Prikl. Mat. Mekh.* Vol. 59, No. 4, pp. 553–561, 1995.

2. *Allowance for specification of constraints imposed on the "uncontrollable" variables.* An alternative to the "worst case" scenario. This approach has various meanings.

2a. *Rationalization of the formulation of the PS problem.* This requires subjecting the system to certain general estimates (possibly including integral estimates) for the "uncontrollable" variables. This considerably simplifies the solution. An example is the study in [10] of the stability of motion of bodies containing cavities filled with liquid.

2b. *"Built-in" possibilities for facilitating the solution.* Put differently: the use of additional relationships linking the components of the phase vector of system (0.1). The feasibility of such relationships must somehow be verified when solving the problem. This approach provided the basis, for example for solving PS problems by constructing auxiliary systems [13].

3. *Allowance for a knowledge (even if only coarse) of estimates for "uncontrollable" variables.* In such cases the PS problem for system (0.1) may be reduced to a problem of stability with respect to all the variables in an auxiliary system of differential equations of the same dimensions [3].

2. EXTENSIONS OF THE MAIN PARTIAL STABILITY THEOREM

Let us suppose that the notion of sign-definiteness with respect to some of the variables has been defined in some specific manner for the Lyapunov V-functions used in the main theorem. This implies that the possible set of variables with respect to which the V-function is sign-definite has been enlarged (by including not only phase variables of system (0.1) but also certain functions of those variables). It is also assumed that the conditions imposed on the "uncontrollable" variables have been clearly specified.

Certain assumptions are made in [10] concerning the continuity of the right-hand side of system (0.1) and the uniqueness and $\mathbf{z}$-continuability of the solutions of the system. In addition, the following functions are introduced: (1) $a(r)$—continuous and monotone increasing for $r \in [0, H]$; (2) a scalar function $V(t, \mathbf{x})$ and a vector-valued function $\mathbf{W}(t, \mathbf{x})$, both continuously differentiable in the domain (1.1). It is assumed that $a(0) = V(t, \mathbf{0}) \equiv \mathbf{0}, \mathbf{W}(t, \mathbf{0}) \equiv \mathbf{0}$. We shall let $\dot{V}$ denote the derivative of V along trajectories of system (0.1).

Theorem 1. Suppose that, given system (0.1), one can find a scalar function $V(t, \mathbf{x})$ and a vector-valued function $\mathbf{W}(t, \mathbf{x})$ such that, in the domain

$$t \geqslant 0, \quad \| \mathbf{y} \| + \| \mathbf{W}(t, \mathbf{x}) \| \leqslant H, \quad \| \mathbf{z} \| < \infty \tag{2.1}$$

the following conditions are satisfied

$$V(t, \mathbf{x}) \geqslant a(\| \mathbf{y} \| + \| \mathbf{W}(t, \mathbf{x}) \|) \tag{2.2}$$

$$\dot{V} \leqslant 0 \tag{2.3}$$

Then the position $\mathbf{x} = \mathbf{0}$ of system (0.1) is $\mathbf{y}$-stable.

The *proof* follows the lines of [2]. For any $\varepsilon > 0, t_0 \geqslant 0, \varepsilon \in (0, H)$, it follows from the continuity of V and a and from the conditions $a(0) = V(t, \mathbf{0}) \equiv 0$ that one can find a number $\delta(\varepsilon, t_0) > 0$ such that, if $\| \mathbf{x}_0 \| < \delta$, then $V(t_0, \mathbf{x}_0) \leqslant a(\varepsilon)$. By (2.2) and (2.3), if $\mathbf{x} = \mathbf{x}(t; t_0, \mathbf{x}_0)$ is a solution such that $\| \mathbf{x}_0 \| < \delta$, then for all $t \geqslant t_0$

$$a(\| \mathbf{y}(t; t_0, \mathbf{x}_0) \| + \| \mathbf{W}(t, \mathbf{x}(t; t_0, \mathbf{x}_0)) \|) \leqslant V(t, \mathbf{x}(t; t_0, \mathbf{x}_0)) \leqslant V(t_0, \mathbf{x}_0) \leqslant a(\varepsilon)$$

By the properties of $a(r)$, this implies $\| \mathbf{y}(t; t_0, \mathbf{x}_0) \| + \| \mathbf{W}(t, \mathbf{x}(t; t_0, \mathbf{x}_0)) \| < \varepsilon, t \geqslant t_0$. Consequently, $\| \mathbf{y}(t; t_0, \mathbf{x}_0) \| < \varepsilon, t \geqslant t_0$, which proves the theorem.

Discussion of Theorem 1. 1. If $\mathbf{W} = \mathbf{0}$, this is Rumyantsev's partial stability theorem [2]. Inequalities (1.1), which represent the "worst case" scenario with regard to the variation of the "uncontrollable" $\mathbf{z}$-variables, are replaced by the more restrictive inequalities (2.1).

2. Inequality (2.2) means that the V-function is sign-definite relative to $\mathbf{y}$ and the components of the $\mathbf{W}$-function, i.e. ($\mathbf{y}$, $\mathbf{W}$)-sign-definite in the sense of [2]. It is not a priori obvious how to choose a suitable $\mathbf{W}$-function; it must be defined when solving the problem. In this sense the $\mathbf{W}$-function plays the part of a second (vector-valued) Lyapunov function (together with the first, scalar-valued, V-function).

3. We know [10] that a V-function, sign-definite with respect to all its variables, need not be sign-definite with respect to only some of them. That is why, even when $\dim(\mathbf{y}) = \dim(\mathbf{z}) = 1$, a function $V(y_1, z_1)$ satisfying conditions

(2.2) and (2.3) may not be sign-definite either in Lyapunov's sense or relative to y_1 (in the sense of [2]). An example is the function [14]

$$V = y_1^2(1+z_1^2)(1+y_1^4 z_1^4)^{-1} \tag{2.4}$$

Indeed, although condition (2.2) holds for $W = y_1 z_1$ and sufficiently small H in the domain (2.1), one has $\lim V = 0$ for any fixed y_1 and $z_1 \to \infty$. Hence the function is sign-definite neither in Lyapunov's sense ($V = 0$ for $y_1 = 0$ and any z_1) nor even in the sense that $V(t, \mathbf{x}) \geqslant a(\| \mathbf{y} \|)$ in the domain (1.1)—the condition for a function to be y-sign-definite [2]. Hence it is not sufficient to choose a V-function to test for the y-stability of the position $\mathbf{x} = \mathbf{0}$ of system (0.1) that is sign-definite relative to y only (or even relative to a larger number of phase variables). One must also consider V-functions that are sign-definite relative to both y and (simultaneously) certain functions $\mathbf{W} = \mathbf{W}(t, \mathbf{x})$. The properties of such V-functions, even in the case dim (y) = dim (z) = 1, may be "intermediate" between sign-definiteness with respect to y [2] and in Lyapunov's sense.

Note that for every $0 < c < 1/2$ the curves defining the level surfaces $V = c$ of the V-function (2.4) fall into two disjoint classes. The first class consists of open curves encircling the axis $y_1 = 0$, which are typical for classical y_1-sign-definite V-functions (in the sense of [2]). The special feature here is the existence of a second, additional class of curves: it, too, consists of open curves, but these curves recede from the position $y_1 = z_1 = 0$ as c is decreased and asymptotically approach the axes $y_1 = 0$ and $z_1 = 0$ as $z_1 \to \infty$ and $y_1 \to \infty$ respectively.

4. If V depends (explicitly) on t, then even when dim (y) = dim (z) = 1 one may have "intermediate" properties not only as $z_1 \to \infty$. For example, the function

$$V = y_1^2(1+e^{2t} z_1^2)(1+e^{4t} y_1^4 z_1^4)^{-1} \tag{2.5}$$

is not sign-definite in Lyapunov's sense. For $W = e^t y_1 z_1$ and sufficiently small H, condition (2.2) holds in the domain (2.1). But this V-function is not y_1-sign-definite in the sense of [2], because $V \to 0$ as $z_1 \to \infty$ (or $t \to \infty$) for any fixed y, y_1 (or y_1, z_1)

5. In the case

$$V(t, \mathbf{x}) \equiv V^*(t, \mathbf{y}, \mathbf{W}(t, \mathbf{x})) \tag{2.6}$$

the verification of condition (2.2) reduces to verifying that V^* is sign-definite in Lyapunov's sense. In particular, if V^* is a quadratic form in the variables y and W, one can use the general Routh–Hurwitz criterion. Incidentally, if the V-function possesses the structure of (2.6) the possibilities for solving the problem posed in [10]—analysis of the y-stability of the position $\mathbf{x} = \mathbf{0}$ of system (0.1) in terms of quadratic forms—are improved considerably. Unlike [10], the structure represented by (2.6) enables "essentially non-linear" V-functions to be used in solving that problem.

6. Theorem 1 may be extended in various directions. Thus, if one demands that, besides conditions (2.2) and (2.3), the condition $V(t, \mathbf{0}, \mathbf{z}) \equiv \mathbf{0}$ should also hold in the domain (2.1), then the position $\mathbf{x} = \mathbf{0}$ of system (0.1) is y-stable for large $\mathbf{z}_0$ (in the sense of [13]).

Let $b(r)$ and $U(t, \mathbf{x})$ be functions of the same type as a and V.

Theorem 2. Suppose that, given system (0.1), one can find two scalar functions $V(t, \mathbf{x})$, $U(t, \mathbf{x})$ and a vector-valued function $\mathbf{W}(t, \mathbf{x})$ such that, besides (2.3), the following conditions also hold in (2.1)

$$V(t, \mathbf{x}) \geqslant a(\| \mathbf{y} \|) \tag{2.7}$$

$$U(t, \mathbf{x}) \geqslant b(\| \mathbf{W}(t, \mathbf{x}) \|), \quad \dot{U} \leqslant 0 \tag{2.8}$$

Then the position $\mathbf{x} = \mathbf{0}$ of system (0.1) is y-stable.

Proof. If conditions (2.2) and (2.7) hold in the domain (2.1) for any $\varepsilon > 0$, $t_0 \geqslant 0$, $\varepsilon \in (0, H')$, one can find a number $\delta_1(\varepsilon, t_0) > 0$ such that, if $\mathbf{x} = \mathbf{x}(t; t_0, \mathbf{x}_0)$ is a solution, $\| \mathbf{x}_0 \| < \delta_1$, satisfying the inequality $\| \mathbf{y}(t; t_0, \mathbf{x}_0) \| + \| \mathbf{W}(t, \mathbf{x}(t, t_0, \mathbf{x}_0)) \| \leqslant H$ for all $t \geqslant t_0$, then $\| \mathbf{y}(t; t_0, \mathbf{x}_0) \| < \varepsilon$, $t \geqslant t_0$.

On the other hand, if condition (2.8) holds in (2.1), then for any $t_0 \geqslant 0$ one can find a number $\delta_2(t_0, \varepsilon) > 0$ such that, if $\| \mathbf{x}_0 \| < \delta_2$, then $\| \mathbf{W}(t, \mathbf{x}(t; t_0, \mathbf{x}_0)) \| \leqslant H - \varepsilon$, $t \geqslant t_0$.

Setting $\delta = \min \{\delta_1, \delta_2\}$, we conclude as a result that for any $\varepsilon > 0$, $t_0 \geqslant 0$, $\varepsilon \in (0, H)$ and every solution $\mathbf{x} = \mathbf{x}(t; t_0, \mathbf{x}_0)$ such that $\| \mathbf{x}_0 \| < \delta$, it is also true that $\| \mathbf{y}(t; t_0, \mathbf{x}_0) \| < \varepsilon$, $t \geqslant t_0$.

Discussion of Theorem 2. 1. A V-function satisfying inequality (2.2) in the domain (2.1) also satisfies inequality (2.7). Consequently, it is y-sign-definite in the domain (2.1) (but not in the domain (1.1), as in [2]). However, the validity of conditions (2.3) and (2.7) in the domain (2.1) does not guarantee the truth of the inequality $\| \mathbf{y}(t; t_0, \mathbf{x}_0) \| + \| \mathbf{W}(t, \mathbf{x}(t; t_0, \mathbf{x}_0)) \| \leqslant H$, $t \geqslant t_0$ along all solutions of system (0.1) for sufficiently small $\| \mathbf{x}_0 \|$. (There is no guarantee for the verification of the constraints (2.1) imposed here on the "uncontrollable" variables.) Therefore, a V-function that satisfies conditions (2.3) and (2.7) in the domain (2.1) does not guarantee **y**-stability of the position $\mathbf{x} = \mathbf{0}$ of system (0.1). But if condition (2.3) holds, a (**y**, **W**)-sign-definite V-function does guarantee **y**-stability. In that case the constraints imposed on the "uncontrollable" variables are indeed observed.

2. A V-function that is **y**-sign-definite in (2.1) guarantees **y**-stability if it can be proved that $\| \mathbf{W}(t, \mathbf{x}) \| \leqslant H, t \geqslant t_0$ along the corresponding solutions of system (0.1). For example, this can be done using one more Lyapunov function. This approach (condition (2.8)) is implemented in Theorem 2.

3. SPECIFICATION OF THE NOTION OF A y-SIGN-DEFINITE V-FUNCTION

One can avoid the situation in which V-functions possesses "intermediate" properties of the kind considered above. To that end the fact that V is (or is not) **y**-sign-definite must be verified not in the domain (1.1) but over the set $M = \{\mathbf{x}: \mathbf{x}(t; t_0, \mathbf{x}_0)\}$ of solutions of system (0.1) for sufficiently small $\| \mathbf{x}_0 \|$. The verification (when possible) need be performed only for **y**-stability.

Theorem 3. Suppose that, given system (0.1), one can find a function $V(t, \mathbf{x})$ such that, besides (2.3), the following condition also holds in M

$$V(t, \mathbf{x}(t; t_0, \mathbf{x}_0)) \geqslant a(\| \mathbf{y}(t; t_0, \mathbf{x}_0) \|) \tag{3.1}$$

Then the position $\mathbf{x} = \mathbf{0}$ of system (0.1) is **y**-stable.

Discussion of Theorem 3. 1. The conditions of Theorem 3 deviate from those adopted in [2] in that **y**-sign-definiteness is verified for V not in the domain (1.1) but over the set M. A similar result (unrelated to the questions under discussion here) was established in [6].

2. If condition (2.3) is satisfied, the V-functions (2.4) and (2.5) will be y_1-positive-definite in the sense of (3.1). Indeed, by (2.3), one has $| W(t; \mathbf{x}(t; t_0, \mathbf{x}_0)) | \leqslant H$ in the set M. Thus, cases that lead to "intermediate" properties are excluded.

3. The function

$$V = (y_1^2 + z_1^2)(1 + z_1^4)^{-1} \tag{3.2}$$

(see [10]) is a sign-definite in Lyapunov's sense but not relative to y_1 (in the sense of [2]). But if $\dot{V} \leqslant 0$, this function is y_1-sign-definite in the sense of (3.1).

Note that for every $0 < c < 1/2$ the curves that define the surfaces $V = c$ of the V-function (3.2) split into two disjoint classes. The first consists of closed curves encircling the position $y_1 = z_1 = 0$, which are typical for classical Lyapunov V-functions. The special feature is the existence of a second, additional class; it consists of open curves that recede from the position $y_1 = z_1 = 0$ as c is decreased.

4. EXAMPLE

Consider the motion of a point of unit mass in a constant gravitational field, constrained to move on the surface

$$x_3 = f(x_1, x_2), \quad f = x_1^2(1 + x_2^2)(1 + x_1^4 x_2^4)^{-1} \tag{4.1}$$

in three-dimensional $x_1x_2x_3$-space, with the x_3-axis pointing vertically upward.

The kinetic and potential energies are

$$T = \frac{1}{2}\left\{ \dot{x}_1^2 + \dot{x}_2^2 + \left[\left(\frac{\partial f}{\partial x_1}\right)\dot{x}_1 + \left(\frac{\partial f}{\partial x_2}\right)\dot{x}_2\right]^2 \right\}$$

$$\Pi = gf(x_1, x_2), \quad g = \text{const} > 0$$

Putting $\mathbf{y} = (x_1, \dot{x}_1, \dot{x}_2)$, $z = x_2$ and introducing auxiliary functions $V = T + \Pi$, $W = x_1x_2$, we obtain

$$V \geqslant \frac{1}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right) + g\left(x_1^2 + W^2\right)\left(1 + W^4\right)^{-1}, \quad \dot{V} \equiv 0 \tag{4.2}$$

Consequently, conditions (2.2) and (2.3) hold in (2.1) for sufficiently small H. As a result, the equilibrium position of the point

$$x_i = x_i = 0 \ (i = 1, 2, 3) \tag{4.3}$$

is y-stable by virtue of Theorem 1.

At the same time, the V-function is not sign-definite—whether relative to y in the sense of [2] ($V \to 0$ for $x_1' = x_2' = 0$, $|x_2| \to \infty$ and any fixed x_1) or in Lyapunov's sense ($V = 0$ for $x_1 = x_2' = x_2'$ and any x_2).

By (4.2), the position (4.3) is also W-stable. Summing up in view of (4.1), we conclude that this position is stable with respect to x_1, x_3, x_2', x_2' (including the case of large x_{20} in the sense of [13]).

In addition, we note that the function $V = T + \Pi$ is y-sign-definite in the sense of (3.1) and satisfies the assumptions of Theorem 3.

5. SPECIAL FEATURES IN THE PERFORMANCE OF PS SYSTEMS

We will now formulate a few assumptions that afford a deeper understanding of the laws governing PS system performance.

1. *The predictability of structural changes as a precondition for normal PS system performance.* This assumption is motivated by the greater sensitivity of the PS property (compared with stability with respect to all variables) to changes in system structure. The implication is that the idea of "robustness" in the PS theory cannot be as general as it is in the theory of stability with respect to all variables. This is natural, because PS theory deals with more "delicate" cases, in which "improved" or "better" stability is simply impossible. In addition, PS properties are sometimes not just desirable but absolutely necessary [14].

Owing to this conclusion, any decision about whether to use the results of PS theory must be made at the design stage in each specific case.

However, if the PS problem is treated as an auxiliary tool in the analysis of stability with respect to all the variables, the situation is different. In such cases PS analysis is admissible within limits determined by the robustness of the property of stability with respect to all variables.

For example, partial stabilization (a development of the PS problem as applied to control systems) plays an auxiliary role in the design of robust control strategies for the angular motion of bodies (such as spacecraft) [13, 14]. This approach has also been extended to game-theoretical control problems under conditions of interference [23, 24].

A better understanding of the problem may also be gained by clarifying the nature of the relationships among notions that determine whether PS properties are preserved. Among the latter are PS properties under persistent perturbations (PP) and parametric perturbations.

2. *The problem of PS under PP is not generally equivalent to the PS problem of preserving stability even under small parametric perturbations.* This is the case even for linear autonomous systems. Such systems, although partially stable under PP, may lose their stability when even slight changes are made in certain coefficients. This does not occur in the problem of stability with respect to all the variables.

At the other end of the "fragility" scale for PS properties one has the following.

3. *A system that loses PS is nevertheless frequently "structurally stable" in the Andronov–Pontryagin sense.* The phase portrait of a "structurally stable" system is, in principle, invariant under minor variations of the parameters. Hence the loss of PS properties in such cases implies only a certain "rotation" of the phase portrait in the phase plane.

4. *The possibility of the invariance of PS properties under arbitrarily large PP* in certain channels of system (0.1) [13]. This question is related to the general problem of invariance [25].

6. EXAMPLES

1. Consider the system

$$A_1 \dot{x}_1 = (A_2 - A_3) x_2 x_3 + u_1 \quad (1\ 2\ 3) \tag{6.1}$$

which describes the angular motion of a rigid body driven by controlling torques u_i ($i = 1, 2, 3$). (Only one of the three equations is shown; the others are obtained by cyclic permutation of the indices $1 \to 2 \to 3$.) In the case $u_i = \alpha_i x_i$ ($\alpha_i = \text{const} < 0, i = 1, 2$), $u_3 \equiv 0$ the equilibrium position $x_i = 0$ ($i = 1, 2, 3$) of the body is asymptotically (x_1, x_2)-stable (property K) for any admissible values of A_i. And then, if A_3 is the greatest or smallest of the A_is, property K holds for any x_{30} (property K_1). If in addition $A_1(A_2)$ is the mean value of A_i, the equilibrium position is also asymptotically $x_1(x_2)$-stable for large $x_{20}(x_{10})$ in the sense of [13].

Property K_1 may be verified by using a Lyapunov function independent of x_3, namely, $V = A_1(A_1 - A_3)x_1^2 + A_2(A_2 - A_3)x_2^2$. Consequently, it is also preserved under PPs $\mathbf{R} = \mathbf{R}(t, \mathbf{x})$, $\mathbf{R}(t, \mathbf{0}) \not\equiv 0$ of the form $\mathbf{R} = (0, 0, R_3)$, i.e. under PPs in one channel of system (6.1). ($R_3(t, \mathbf{x})$ is any function such that the conditions for existence, uniqueness and x_3-continuability of solutions of the "perturbed" system (6.1) hold in a domain $t \geqslant 0, |x_i| \leqslant H$ $(i = 1, 2), |x_3| < \infty$.) Thus, property K_1 is invariant under PPs of the type described.

Cases are possible in which a PS property is not only preserved but also becomes asymptotic when PPs are applied (cf. examples in ([13]). In these cases, however, further restrictions must be imposed on the PPs.

2. The position $y_1 = z_1 = 0$ of the system

$$y_1^{\cdot} = -y_1 + \varepsilon z_1, \quad z_1^{\cdot} = z_1 \quad (\varepsilon = \text{const}) \tag{6.2}$$

is asymptotically y_1-stable only if $\varepsilon = 0$. On the other hand, this is a system of "general position" [26] for any ε, with a phase portrait of the "saddle" type. As ε passes through the value $\varepsilon = 0$, only a "rotation" of the phase portrait is observed. The rotation can be as small as desired provided ε is sufficiently small. When $\varepsilon = 0$, the separatrices of the "saddle" coincide with the coordinate axes of the phase plane of system (6.2).

7. MECHANISMS GIVING RISE TO PS PROPERTIES

The "degree of structural stability" of a PS system depends on the mechanism by which PS properties arise. We will dwell on one aspect of this problem.

PS properties may be produced by different mechanisms even in linear systems. Thus, consider the mechanism giving rise to asymptotic y-stability in the following linear autonomous system, which is unstable in Lyapunov's sense

$$\mathbf{y}^{\cdot} = A\mathbf{y} + B\mathbf{z}, \quad \mathbf{z}^{\cdot} = C\mathbf{y} + D\mathbf{z} \tag{7.1}$$

$$\mathbf{y} \in R^m, \quad \mathbf{z} \in R^p, \quad m \geqslant 1, \quad p \geqslant 2$$

Here the terms of $\mathbf{z}(t)$-solutions with non-negative eigenvalues are cancelled out in the linear combinations $B\mathbf{z}$ of "uncontrollable" z-variables. This is a stiff mechanism. How it "works" depends on the set of numbers of the matrices B and D. That is why the conditions for asymptotic y-stability in the unstable system (7.1) may include not only inequalities but also equalities linking the coefficients of the system.

However, when one considers non-autonomous systems—even linear, and *a fortiori* non-linear ones—the mechanism by which PS properties arise is "softened". Even in linear non-autonomous systems, PS criteria including no equalities among the system coefficients are quite normal.

In particular [13], one mechanism is based on "compensation" in $B(t)\mathbf{z}$ for the undesirable tendency $\|\mathbf{z}\| \to \infty$ in the functions occurring in $B(t)$. Other mechanisms producing PS properties are possible in linear systems. They are more deeply related to the process of integrating such systems.

8. RELATIONSHIPS BETWEEN CONDITIONS FOR y-STABILITY AND z-CONTINUABILITY OF SOLUTIONS OF SYSTEM (0.1)

Solutions of system (0.1) may not be z-continuable even if they are y-stable. For that reason z-continuability and y-stability are generally studied separately. In fact, y-stability conditions in themselves may enable systems with z-non-continuable solutions to "pass" as admissible.

For example, consider the system

$$y_1^{\cdot} = -y_1 + y_1^2 z_1, \quad z_1^{\cdot} = z_1 - 2y_1 z_1^2 \tag{8.1}$$

The Lyapunov function $V = y_1^2 + (-y_1 + y_1^2 z_1)^2 \geqslant y_1^2$, $\dot{V} \leqslant 0$ satisfies the assumptions of the theorem proved in [2] concerning the y_1-stability of the position $y_1 = z_1 = 0$. The assumptions of the asymptotic y_1-stability theorem [13] are also satisfied, since system (8.1) admits of the construction of an auxiliary $\boldsymbol{\mu}$-system [13] $\dot{y}_1 = -y_1 + \mu_1$, $\dot{\mu}_1 = -\mu_1$.

However, the solutions of system (8.1) (assuming, for simplicity, that $t_0 = 0$) have the form

$$V = y_1^2 + (-y_1 + y_1^2 z_1)^2 \geqslant y_1^2, \quad V^{\cdot} = \leqslant 0$$

Consequently, they are not z_1-continuable. In an arbitrarily small neighbourhood of $y_1 = z_1 = 0$ there are fixed values $y_{10}, z_{10}, (y_{10}z_{10} < 0)$ such that $z_1 = \infty$ for $t = -(y_{10}z_{10})^{-1} > 0$.

Thus, both the method of Lyapunov functions and the method of $\boldsymbol{\mu}$-systems [13] in *PS* problems may enable systems with z-non-continuable solutions to "pass" as admissible.

I wish to thank V. V. Rumyantsev for discussions on this article.

This research was carried out with the financial support of the Russian Foundation for Basic Research (95-01-00506a), the International Science Foundation and the Government of the Russian Federation (JJS100).

REFERENCES

1. LYAPUNOV A. M., Investigation of one of the special cases of the problem of stability of motion. In *Collected Papers*, Vol. 2, pp. 272–331. Izd. Akad. Nauk SSSR, Moscow, 1956.
2. RUMYANTSEV V. V., On the stability of motion relative to some of the variables. *Vestnik Mosk. Gos. Univ., Ser. Mat. Mekh. Fiz. Astron., Khim*, 4, 9–16, 1957.
3. ZUBOV V. I., *Mathematical Methods of Investigating Automatic Control Problems*. Sudpromgiz, Leningrad, 1959.
4. CORDUNEANU C., Sur la stabilité partielle. *Rev. Roum. Math. Pure et Appl.* **9**, 3, 229–236, 1964.
5. MATROSOV V. M., Development of the method of Lyapunov functions in the theory of stability of motion. In *Proc. 2nd All-Union Conference on Theoretical and Applied Mechanics*, Vol. 1, pp. 112–125. Nauka, Moscow, 1965.
6. HALANAY A., *Differential Equations: Stability, Oscillations, Time Lags*. Academic Press, New York, 1966.
7. PFEIFFER K. and ROUCHE N., Lyapunov's second method applied to partial stability. *J. Méc.* **8**, 2, 323–334, 1969.
8. RISITO C., Sulla stabilita asintotica parzialle. *Ann. Math. Pura et Appl.* **84**, 279–292, 1970.
9. FERGOLA P. and MOAURO V., On partial stability. *Ric. Mat.* **19**, 2, 185–207, 1970.
10. RUMYANTSEV V. V. and OZIRANER A. S., *Stability and Stabilization of Motion Relative to Some of the Variables*. Nauka, Moscow, 1987.
11. HATVANI L., On the stability of the solutions of ordinary differential equations with mechanical applications. *Alkalm. Mat. Lap.* **15**, 1/2, 1–90, 1990/1991.
12. ANDREYEV A. S., The investigation of partial asymptotic stability. *Prikl. Mat. Mekh.* **55**, 4, 539–547, 1991.
13. VOROTNIKOV V. I., *Stability of Dynamical Systems with Respect to Some of the Variables*. Nauka, Moscow, 1991.
14. VOROTNIKOV V. I., Problems and methods of investigating the stability and stabilization of motion relative to some of the variables: research trends, results, special features. *Avtom. Telemekh.* 3, 3–62, 1993.
15. KOZLOV V. V., Stability of periodic trajectories and Chebyshev polynomials. *Vestnik Mosk. Gos. Univ., Ser. 1. Matem. Mekh.* 5, 7–13, 1991.
16. VUJIČIĆ V. A. and KOZLOV V. V., Lyapunov stability with respect to given state functions. *Prikl. Mat. Mekh.* **55**, 4, 555–559, 1991.
17. MOVCHAN A. A., Stability of processes with respect to two metrics. *Prikl. Mat. Mekh.* **24**, 6, 988–1001, 1960.
18. LAKSHMIKANTHAM V. and SALVADORI L., On Massera type converse theorem in terms of two different measures. *Boll. Unione Mat. Ital. Ser. A* **13**, 2, 293–301, 1976.
19. SIRAZETDINOV T. K., *The Stability of Systems with Distributed Parameters*. Nauka, Novosibirsk, 1987.
20. AMINOV A. B. and SIRAZETDINOV T. K., The method of Lyapunov functions in problems of polystability of motion. *Prikl. Mat. Mekh.* **51**, 5, 709–716, 1987.
21. MARTYNYUK A. A., A theorem on polystability. *Dokl. Akad. Nauk SSSR* **318**, 4, 808–811, 1991.
22. KRASOVSKII N. N., *Control of a Dynamical System. Problem of the Minimum of the Guaranteed Result.* Nauka, Moscow, 1985.
23. VOROTNIKOV V. I., The control of the angular motion of a rigid body. Game-theoretic approach. *Prikl. Mat. Mekh.* **58**, 3, 82–103, 1994.
24. VOROTNIKOV V. I., The problem of the non-linear synthesis of time-suboptimal bounded controls under interference. *Izv. Ross. Akad. Nauk, Tekhn. Kibern.* 6, 88–116, 1994.
25. KUKHTENKO A. I., Basic steps in the creation of a theory of invariance, I, II. *Avtomatika* 2, 3–13, 1984; 2, 3–14, 1985.
26. ARNOL'D V. I., *Catastrophe Theory*. Nauka, Moscow, 1990.

Translated by D.L.